Lectures on quantum computation by David Deutsch

Lecture 3: Measurement

Worked Examples

1. Define *measurement* and *perfect measurement* in words.

2. Let $\hat{A}$ and $\hat{B}$ be observables.
   (a) Prove that $\hat{A}+\hat{B}$ is an observable.
   (b) Prove that a necessary and sufficient condition for $\hat{A}\hat{B}$ to be an observable is that it equal $\hat{B}\hat{A}$.
   (c) Prove that it is not always the case that to measure $\hat{A}+\hat{B}$, one can measure $\hat{A}$ and $\hat{B}$ and then add the outcomes.
   (d) Prove that if $\hat{A}\hat{B}$ is an observable, there exists a way of measuring it that consists of measuring $\hat{A}$, measuring $\hat{B}$, and then multiplying the outcomes together.

   Note: In fact, though we have not proved it here, if $\hat{A}\hat{B}$ is an observable, *any* perfect measurements of $\hat{A}$ and $\hat{B}$, followed by multiplying the outcomes together, constitutes a perfect measurement of $\hat{A}\hat{B}$.

3. We shall prove in the Worked Examples of Lecture 5 that despite 2(c) above, $\left\langle\left(\hat{A}-\hat{B}\right)^2\right\rangle = 0$ implies that $\hat{A}$ and $\hat{B}$ have the same value in all universes. On that assumption, obtain a necessary condition that a process occurring during the period $0 < t < 1$ constitutes a perfect measurement of $\hat{A}(0)$.

4. Verify the following properties of the outer product:
   (a) The outer product of two Hermitian matrices is Hermitian.
   (b) The outer product operation is associative.
   (c) The outer product operation is not commutative.
   (d) $(A\times B)(C\times D) = (AC)\times(BD)$ (assuming that both sides exist).

5. Verify that the matrices $X = \mathrm{I}\times\sigma_x$, $Y = \mathrm{I}\times\sigma_y$, $Z = \mathrm{I}\times\sigma_z$ (with the unit matrix $\mathrm{I}\times\mathrm{I}$) jointly have the same algebra as the standard observables $\hat{X}(t)$, $\hat{Y}(t)$, $\hat{Z}(t)$ of a qubit at any time t.

6. Suppose that two qubits are in a state such that at a given time, the *Z*-observable of the second qubit is sharp at the value 1. Show that if $\hat{A}_1$ is an arbitrary observable of the first qubit at that time, $\left\langle \hat{A}_1\hat{Z}_2\right\rangle = \left\langle \hat{A}_1\right\rangle$.

7. If $\hat{A}$ is a Boolean observable with spectrum $\{a,b\}$, show that

$$\left|\frac{2\langle\hat{A}\rangle - a - b}{a-b}\right|$$

is a measure of the sharpness of $\hat{A}$ that depends only on the probabilities of the outcomes of a measurement of $\hat{A}$, not on the values *a* and *b* themselves.

8. Between times $t=0$ and $t=1$ a perfect measurement of the observable $\hat{Z}_1(0)$ of qubit 1 takes place, and the outcome is stored in the observable $\hat{Z}_2(1)$ of qubit 2. Given that a controlled-not gate has the following dynamics:

$$\begin{array}{l|l} \hat{X}_1(t+1)=\hat{X}_1(t)\hat{X}_2(t) & \hat{X}_2(t+1)=\hat{X}_2(t) \\ \hat{Y}_1(t+1)=\hat{Y}_1(t)\hat{X}_2(t) & \hat{Y}_2(t+1)=\hat{Z}_1(t)\hat{Y}_2(t) \\ \hat{Z}_1(t+1)=\hat{Z}_1(t) & \hat{Z}_2(t+1)=\hat{Z}_1(t)\hat{Z}_2(t) \end{array}$$

show, using the sharpness measure of Example 7, that

(a) $\hat{Z}_1(1)$ is exactly as sharp or un-sharp as $\hat{Z}_1(0)$.
(b) $\hat{Z}_2(1)$ is exactly as sharp or un-sharp as $\hat{Z}_1(0)$ and $\hat{Z}_1(1)$.

9. Suppose that the measurement of Example 8 is performed in a state where $\hat{Z}_1$ is initially un-sharp, so that neither $\hat{Z}_1$ nor $\hat{Z}_2$ is sharp at the end of the measurement:

(a) Show that $\hat{Z}_1\hat{Z}_2$ is un-sharp at the beginning of the measurement.
(b) Show that $\hat{Z}_1\hat{Z}_2$ is sharp at the and of the measurement.

10. Suppose that in the interference experiment analysed in Lecture 2, the $\hat{Z}$ observable of the qubit is measured just after the photon passes through the first beam splitter. (In other words, we measure which path the photon travels on.) Show that the interference phenomenon no longer happens, and show that at the end of such an experiment, no Boolean observable of either qubit is sharp.

Hints

1. (Define measurement and perfect measurement.) Measurements are physical processes.

2. ($\hat{A}+\hat{B}$ and $\hat{A}\hat{B}$.)
   (a) Observables correspond to Hermitian matrices.
   (b) See 2(a).
   (c) Consider the eigenvalues of those observables.
   (d) Note that if $\hat{A}$ and $\hat{B}$ commute, then they and $\hat{A}\hat{B}$ can all be expressed as functions of a single observable.

3. (Condition for a perfect measurement.) Translate the answer to Example 1 into mathematical form.

4. (Properties of the outer product.) Let matrix *B* have dimension *m*, and let its indices run from 0 to $m-1$. By the definition given in the Lecture, the (a,b)'th element of the matrix $A\times B$ is

$$(A\times B)_{ab} = A_{\lfloor a/m\rfloor,\lfloor b/m\rfloor}B_{a \bmod m, b \bmod m} .$$

5. (The matrices $X = \mathrm{I}\times\sigma_x$, $Y = \mathrm{I}\times\sigma_y$, $Z = \mathrm{I}\times\sigma_z$.) Use 4(d).

6. (*Z*-observable of the second qubit is sharp at the value 1.) $\hat{A}_1$ and $\hat{Z}_2$ commute because they are observables of different systems at the same time. Hence the result of 2(d) applies.

7. (Measure of sharpness.) Express the expectation value of $\hat{A}$ in terms of its eigenvalues *a* and *b* and the probability *p* that the outcome of a measurement of $\hat{A}$ will be *a*. $\hat{A}$ is maximally sharp when $p=1$ or $p=0$ and minimally sharp when $p=\frac{1}{2}$.

8. (Perfect measurement propagates un-sharpness.) One can regard a perfect measurement of a Boolean observable as the computation performed by a controlled-not gate. Use the result of Example 6,

9. (Sharp observable at the end of a measurement of an un-sharp observable.) Use the dynamical equations and the result of Example 6,

10. (Effect of measuring a qubit halfway through an interference phenomenon.) Use the dynamics of the beam splitter and ***not*** gate, and the definition of interference, as in the Worked Examples for Lecture 2, and regard the measurement process as the computation performed by a controlled-not gate.

Answers

1. (Define measurement and perfect measurement.)

   A measurement is a physical process in which an observable of one system at one time comes to depend on an observable of another system at an earlier time.

   A perfect measurement is defined by its effect in states in which the observable being measured is sharp: it records sharp values accurately, and it leaves them unchanged.

2. ($\hat{A}+\hat{B}$ and $\hat{A}\hat{B}$.)

   (a) Consider the smallest system of which $\hat{A}$ and $\hat{B}$ are both observables. There is a 1-1 correspondence between the set of all observables of that system and the set of all Hermitian matrices of a certain dimension. If $\hat{A}$ and $\hat{B}$ are each Hermitian with that dimension, than so is $\hat{A}+\hat{B}$, so it must be an observable of the same system.

   (b) The Hermitian conjugate of $\hat{A}\hat{B}$ is $\left(\hat{A}\hat{B}\right)^{\dagger}=\hat{B}^{\dagger}\hat{A}^{\dagger}=\hat{B}\hat{A}$, and so $\hat{A}\hat{B}$ is Hermitian if and only if $\hat{A}\hat{B}=\hat{B}\hat{A}$ i.e. if $\hat{A}$ and $\hat{B}$ commute. Hence $\hat{A}\hat{B}$ is an observable under the same condition.

   (c) Suppose that $\hat{A}=\hat{X}$ and $\hat{B}=2\hat{Y}$ where $\hat{X}$ and $\hat{Y}$ are two of the standard observables of a qubit. Then $\hat{A}$ and $\hat{B}$ have eigenvalues ±1 and ±2 respectively. In a state where $\left\langle\hat{A}\right\rangle=+1$, $\hat{A}$ is sharp and $\hat{B}$ is un-sharp, so the result of measuring $\hat{A}$ and then $\hat{B}$ and adding the results can be either -1 or 3. Similarly, in a state where $\left\langle\hat{A}\right\rangle=-1$, the result of such a process can be either 1 or -3. Hence the process as a whole, if it could be regarded as a measurement of a single observable, would have four possible outcomes and so the observable would have to have at least four eigenvalues. But $\hat{A}+\hat{B}$, being an observable of a qubit, cannot have more than two eigenvalues. Hence that process does not constitute a measurement of it.

   (d) If $\hat{A}\hat{B}$ is an observable then, from 2(b), $\hat{A}$ and $\hat{B}$ commute. Therefore they are diagonal in the same matrix representation. In that representation, let $\hat{A}=\mathrm{diag}\left(a(1),a(2),\ldots,a(n)\right)$ and $\hat{B}=\mathrm{diag}\left(b(1),b(2),\ldots,b(n)\right)$, where *a* and *b* are real-valued functions, and define $\hat{C}=\mathrm{diag}\left(1,2,\ldots,n\right)$. Evidently $\hat{A}=a(\hat{C})$, $\hat{B}=b(\hat{C})$ and $\hat{A}\hat{B}=f(\hat{C})$ where $f(x)\equiv a(x)b(x)$. Hence one way to measure $\hat{A}\hat{B}$ is to measure $\hat{C}$ and then compute the function *f* of the outcome. And one way of doing that is to measure $\hat{C}$ and to compute the function *a* of the outcome (the process so far constituting a measurement of $\hat{A}$), then to compute the function *b* of the outcome (thus completing a measurement of $\hat{B}$), and then to multiply the results of those two processes.

3. Translating the answer to Example 1, using the given assumption, we obtain the following necessary condition for a process to be a perfect measurement of $\hat{A}(0)$ during the period $0<t<1$:

There exists an observable $\hat{B}$ such that whatever $\hat{A}(0)$ is,

$\left\langle \left(\hat{A}(1)-\hat{A}(0)\right)^2 \right\rangle = 0$ and $\left\langle \left(\hat{B}(1)-\hat{A}(0)\right)^2 \right\rangle = 0$.

4. (Properties of the outer product.) Let matrix B have dimension m, and let its indices run from 0 to $m-1$. By the definition given in the Lecture, the (a,b)'th element of the matrix $A \times B$ is

$$\left(AB\right)_{ab} = A_{\lfloor a/m \rfloor, \lfloor b/m \rfloor} B_{a \bmod m, b \bmod m}.$$

(a)
$$\begin{aligned}\left(\left(A \times B\right)^{\dagger}\right)_{ab} &= A^{*}_{\lfloor b/m \rfloor, \lfloor a/m \rfloor} B^{*}_{b \bmod m, a \bmod m} \\ &= A_{\lfloor a/m \rfloor, \lfloor b/m \rfloor} B_{a \bmod m, b \bmod m} \\ &= \left(A \times B\right)_{ab}\end{aligned}$$

(b)
$$\begin{aligned}\left(\left(A \times B\right) \times C\right)_{ab} &= \left(A \times B\right)_{\lfloor a/n \rfloor, \lfloor b/n \rfloor} C_{a \bmod n, b \bmod n} \\ &= A_{\lfloor \lfloor a/n \rfloor/m \rfloor, \lfloor \lfloor b/n \rfloor/m \rfloor} B_{\lfloor a/n \rfloor \bmod m, \lfloor b/n \rfloor \bmod m} C_{a \bmod n, b \bmod n}. \\ \left(A \times \left(B \times C\right)\right)_{ab} &= A_{\lfloor a/mn \rfloor, \lfloor b/mn \rfloor} \left(B \times C\right)_{a \bmod mn, b \bmod mn} \\ &= A_{\lfloor a/mn \rfloor, \lfloor b/mn \rfloor} B_{\lfloor (a \bmod mn)/n \rfloor, \lfloor (b \bmod mn)/n \rfloor} C_{(a \bmod mn) \bmod n, (b \bmod mn) \bmod n},\end{aligned}$$

Which are equal because of the following identities:

$\lfloor \lfloor a/n \rfloor / m \rfloor \equiv \lfloor a/mn \rfloor$

(Proof: both sides are integers between a/mn and $a/mn + 1 - 1/mn$.)

$\lfloor a/n \rfloor \bmod m \equiv \lfloor a \bmod mn/n \rfloor$

(Proof: by induction. As a increases, both sides increase when $n|a$.)

$a \bmod n \equiv \left(a \bmod mn\right) \bmod n$

(Proof: from the properties of modulo.)

(c) $\sigma_x \times \mathrm{I} = \mathrm{diag}(1,1,-1,-1) \quad \neq \quad \mathrm{I} \times \sigma_x = \mathrm{diag}(1,-1,1,-1).$

(d) If both sides exist, the second index of A must have the same range as the first index of C, and the same for B and D.

$$
\begin{aligned}
\left((A\times B)(C\times D)\right)_{ac} &= \textstyle\sum_b (A\times B)_{ab}(C\times D)_{bc} \\
&= \textstyle\sum_b A_{\lfloor a/m\rfloor,\lfloor b/m\rfloor} B_{a \bmod m, b \bmod m} C_{\lfloor b/n\rfloor,\lfloor c/n\rfloor} D_{b \bmod n, c \bmod n} \\
\left((AC)\times(BD)\right)_{ac} &= (AC)_{\lfloor a/m\rfloor,\lfloor c/m\rfloor}(BD)_{a \bmod m, c \bmod m} \\
&= \textstyle\sum_{de} A_{\lfloor a/m\rfloor,d} C_{d,\lfloor c/m\rfloor} B_{a \bmod m, e} D_{e, c \bmod m},
\end{aligned}
$$

and these become identical if we replace the double sum over d and e by a single sum over b where $d = \lfloor b/m \rfloor$ and $e = b \bmod m$.

5. (The matrices $X = \mathrm{I}\times\sigma_x$, $Y = \mathrm{I}\times\sigma_y$, $Z = \mathrm{I}\times\sigma_z$.) Using 4(d):

$$
\begin{aligned}
X^2 &= \mathrm{I}\times{\sigma_x}^2 = \mathrm{I}\times\mathrm{I}; \\
XY &= \mathrm{I}\times\sigma_x\sigma_y = i\,\mathrm{I}\times\sigma_z = iZ;
\end{aligned}
$$

and so on.

6. (Z-observable of the second qubit is sharp at the value 1.) $\hat{A}_1$ and $\hat{Z}_2$ commute because they are observables of different systems at the same time. Hence the result of 2(d) applies, and it is possible to measure $\hat{A}_1\hat{Z}_2$ by measuring $\hat{A}_1$ and $\hat{Z}_2$ separately and then multiplying the outcomes. But since $\hat{Z}_2$ is sharp with the value 1, this product necessarily equals the outcome of measuring $\hat{A}_1$. Hence the expectation value of $\hat{A}_1\hat{Z}_2$ must be equal to that of $\hat{A}_1$.

7. (Measure of sharpness.) Let p be the probability that the outcome of a measurement of $\hat{A}$ would be a. Then $\langle\hat{A}\rangle = pa + (a-p)b$. $\hat{A}$ is perfectly sharp when $p = 1$ or $p = 0$ and minimally sharp when $p = \frac{1}{2}$. The stated measure equals $|2p-1|$, which is independent of a and b, takes its maximum value of 1 when $p = 1$ or $p = 0$ and its minimum value of 0 when $p = \frac{1}{2}$.

8. (Perfect measurement propagates un-sharpness.) One can regard a perfect measurement of a Boolean observable as a computation performed by a controlled-not gate with the target qubit $\hat{Z}_2$ initially sharp at the value 1.
   (a) From the dynamical equations, $\hat{Z}_1(1) = \hat{Z}_1(0)$ and so they must be equally sharp.
   (b) From the dynamical equations, $\hat{Z}_2(1) = \hat{Z}_1(0)\hat{Z}_2(0)$, and so, from the result of Example 6, $\langle\hat{Z}_2(1)\rangle = \langle\hat{Z}_1(0)\rangle$. Hence, by the measure of Example 7, they are equally sharp.

9. (Sharp observable at the end of a measurement of an un-sharp observable.)

(a) Since $\hat{Z}_1$ is initially un-sharp and $\hat{Z}_2$ is initially sharp at the value 1, the conditions of Example 6 apply and the product observable $\hat{Z}_1\hat{Z}_2$ is un-sharp at that time.

(b) From the dynamical equations given in Example 8, $\hat{Z}_1(1)\hat{Z}_2(1) = \hat{Z}_1^{\,2}(0)\hat{Z}_2(0) = \hat{Z}_2(0)$. Hence the final expectation value of $\hat{Z}_1\hat{Z}_2$ is the same as the initial expectation value of $\hat{Z}_2$, namely 1, and the measurement process makes $\hat{Z}_1\hat{Z}_2$ become sharp with that value.

10. (Effect of measuring a qubit halfway through an interference phenomenon.) Let qubit 1 be the one associated with the path of the photon, and qubit 2 be the one in which the outcome of the measurement is stored. Qubit 1 starts in a state where $\left\langle \hat{Z}_1(0) \right\rangle = 1$, as in the unimpeded interference experiment analysed in the Worked Examples for Lecture 2, and qubit 2 likewise starts in the blank state $\left\langle \hat{Z}_2(0) \right\rangle = 1$. Hence, as before, we also have $\left\langle \hat{X}_1(0) \right\rangle = \left\langle \hat{Y}_1(0) \right\rangle = \left\langle \hat{X}_2(0) \right\rangle = \left\langle \hat{Y}_2(0) \right\rangle = 0$. The equations of motion for the beam splitter are

$$\begin{aligned} \hat{X}_1(t+1) &= -\hat{Z}_1(t) \\ \hat{Y}_1(t+1) &= \hat{Y}_1(t) \\ \hat{Z}_1(t+1) &= \hat{X}_1(t) \end{aligned}$$

and the second qubit remains unaffected while the first qubit is passing through it, and so we have:

$$\begin{array}{l|l} \hat{X}_1(1) = -\hat{Z}_1(0) & \hat{X}_2(1) = \hat{X}_2(0) \\ \hat{Y}_1(1) = \hat{Y}_1(0) & \hat{Y}_2(1) = \hat{Y}_2(0) \\ \hat{Z}_1(1) = \hat{X}_1(0) & \hat{Z}_2(1) = \hat{Z}_2(0) \end{array}$$

If the measurement occurs between times 1 and 2, we have (from the controlled-not dynamics given in Example 8):

$$\begin{array}{l|l} \hat{X}_1(2) = -\hat{Z}_1(0)\hat{X}_2(0) & \hat{X}_2(2) = \hat{X}_2(0) \\ \hat{Y}_1(2) = \hat{Y}_1(0)\hat{X}_2(0) & \hat{Y}_2(2) = \hat{X}_1(0)\hat{Y}_2(0) \\ \hat{Z}_1(2) = \hat{X}_1(0) & \hat{Z}_2(2) = \hat{X}_1(0)\hat{Z}_2(0) \end{array}$$

Next (between times 2 and 3), the ***not*** gate acts on qubit 1, with dynamics

$$\begin{aligned} \hat{X}(t+1) &= -\hat{X}(t) \\ \hat{Y}(t+1) &= \hat{Y}(t) \\ \hat{Z}(t+1) &= -\hat{Z}(t) \end{aligned}$$

and so

$$\begin{array}{l|l} \hat{X}_1(3) = \hat{Z}_1(0)\hat{X}_2(0) & \hat{X}_2(3) = \hat{X}_2(0) \\ \hat{Y}_1(3) = \hat{Y}_1(0)\hat{X}_2(0) & \hat{Y}_2(3) = \hat{X}_1(0)\hat{Y}_2(0) \\ \hat{Z}_1(3) = -\hat{X}_1(0) & \hat{Z}_2(3) = \hat{X}_1(0)\hat{Z}_2(0) \end{array}$$

Finally (between times 3 and 4), the second beam splitter acts on the first qubit. Hence

$$\begin{array}{l|l} \hat{X}_1(4) = \hat{X}_1(0) & \hat{X}_2(4) = \hat{X}_2(0) \\ \hat{Y}_1(4) = \hat{Y}_1(0)\hat{X}_2(0) & \hat{Y}_2(4) = \hat{X}_1(0)\hat{Y}_2(0) \\ \hat{Z}_1(4) = \hat{Z}_1(0)\hat{X}_2(0) & \hat{Z}_2(4) = \hat{X}_1(0)\hat{Z}_2(0) \end{array}$$

from which it follows (again from the result of Example 6) that $\hat{Z}_1$ is maximally un-sharp at the end of the experiment, and so the interference phenomenon has not happened.

Likewise, $\hat{X}_1$ and $\hat{Y}_1$ and the three standard observables of qubit 2 are all maximally un-sharp at the end of the experiment. And that implies that no Boolean observable of either qubit is sharp at that time, because the expectation value of any linear combination of the standard observables is then zero too, and by the measure 7, adding any multiple of the unit observable to an observable does not change its sharpness.