Gaussian Operations and Entanglement Distillation

Géza Giedke

Barbara Kraus

Klemens Hammerer

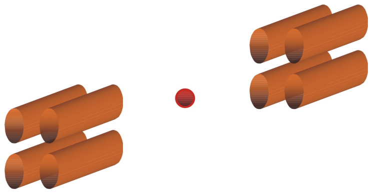
J. Ignacio Cirac

Max-Planck–Institut für Quantenoptik, Garching

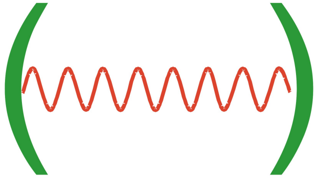
Continuous Variables

many physical systems described by continuous variables (CV)

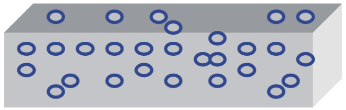
- particle in harmonic trap, 1-d motion
  Hilbert space $\mathcal{H} = \mathrm{L}^2(\mathbb{R})$,
  dynamical variables X, P: $[X, P] = i$



- mode of the optical field
  $\mathcal{H} = \mathrm{span}\{|0\rangle, |1\rangle, \ldots\} \sim \mathrm{L}^2(\mathbb{R})$
  field quadratures X, P



- atomic ensembles (symmetric states)
  collective internal spin: $\frac{1}{\sqrt{N}} J_{x,y} \to X, P$
  $\mathcal{H} = \left(\mathbb{C}^2\right)^N \longrightarrow \mathrm{L}^2(\mathbb{R})$



Continuous Variables for QIP

- have been used for several QIP tasks:
  - on-demand entanglement [Julsgaard et al.; 2000; Silberhorn et al., 2001],
  - teleportation [Furusawa et al., 1998; Kuzmich et al. 2000; Bowen et al. 2002],
  - quantum cryptography [Ou et al., 1992; Silberhorn et al., 2002]
  - interface light-atoms [Schori et al. 2002]
- universal quantum computing with CV: [Lloyd and Braunstein, 1998]
  need Hamiltonians linear, quadratic, **and cubic** in X, P
- **but:** cubic interaction very hard to realize

? **what can be done with limited set of operations?**

Feasible Operations

Case 1: **linear optics**: all quadratic Hamiltonians can be realized:

- passive linear optics: beam splitters, phase plates
- active linear optics: squeezers
- noise: losses, discarding subsystems, thermal fields
- measurements: homodyne detection

Feasible Operations

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Case 2: **atomic ensembles interacting with light**

- interaction Hamiltonian $H_{\text{int}} = X_1 X_2$
- local rotations (no squeezing)
- measurements on the light field

? what is possible under these limitations?

Questions

- can we perform interesting tasks?
  - teleportation, cryptography: yes
  - entanglement distillation ?
  - state transformation $\rho \longrightarrow \rho'$?
  - optimal entanglement generation
  - optimal simulation of desired evolution

- **need to characterize all possible operations**

Questions

pick two:

- entanglement distillation of Gaussian states with linear optics
  - $\rho_{AB}^{\otimes N} \xrightarrow{\text{LOCC}} |\Psi\rangle_{AB} \approx$ maximally entangled
  - important for long-distance quantum communication (repeater)
  - for ρ Gaussian in principle possible with cubic interaction [Duan et al. 2001]
  - many failures to find distillation protocol with linear optics
  - Eisert et al. [2002]: cannot distill 2 copies of 1×1 Gaussian states with (subset of) linear optics LOCC
- optimal entanglement generation with H_{int}
  - entanglement is resource for many applications
  - many results on what can be done: entanglement, spin squeezing, quantum memory [Polzik, Mølmer, Wiseman, Duan and others]
  - make **most efficient use** of precious interaction

Outline

- Gaussian operations [Giedke and Cirac, quant-ph/0204085]
  - allow all the tools of linear optics
  - characterize all Gaussian operations mathematically
  - all Gaussian operations feasible with linear optics
- Distilling Gaussian states with Gaussian operations?
  - characterize Gaussian LOCC
  - distillation **not possible** with Gaussian operations
- interaction $X_1 X_2$ [Giedke, Hammerer, Kraus, and Cirac, quant-ph/0209xy]
  - what time-evolutions are accessible?
  - optimal creation of entanglement

Gaussian States

- typical initial states: vacuum, coherent state, thermal state
- are all **Gaussian states** $\Leftrightarrow$ Wigner function is Gaussian
- Gaussian property preserved by quadratic Hamiltonians
- **correlation matrix** $\gamma \geq iJ \in M_{2n}$, **displacement** $d \in \mathbb{R}^{2n}$

where $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \oplus \ldots$ (symplectic matrix)

- $\gamma_{kl} = 2\mathrm{tr}[(R_k - d_k)(R_l - d_l)\rho] - [R_k, R_l], \quad d_k = \mathrm{tr}(\rho R_k)$

where $\mathbf{R} = (X_{A1}, P_{A1}, X_{A2}, \ldots, X_{B1}, P_{B1}, \ldots), [X_k, P_l] = i\delta_{kl}$.

- for bipartite states: all nonlocal properties contained in CM γ

$$\gamma = \begin{pmatrix} A & C \\ C^T & B \end{pmatrix}, \qquad \Rightarrow \text{take} \quad d = 0$$

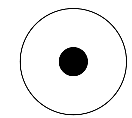
Gaussian Operations

- Gaussian Operation (GO): completely positive map G

$$G : \rho_{\text{(Gaussian)}} \longrightarrow \rho'_{\text{(Gaussian)}}$$

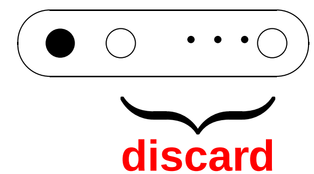
- **Examples are**

unitaries $\gamma \mapsto S^T \gamma S$



$\rho \mapsto U_S^\dagger \rho U_S$

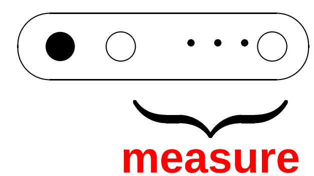
unitaries+ancillas $\gamma \mapsto M^T \gamma M + G$



$\mathrm{tr}_{anc}(U_S^\dagger \rho \otimes \rho_{anc} U_S)$

[Demoen *et al.* 1977]

unitaries + POVMs **???**



$P_\psi U_S^\dagger \rho \otimes \rho_{anc} U_S P_\psi$

$\Rightarrow$

- give **general form** of GO G
- show how to implement map G: is linear optics enough?
- on bipartite systems: locally implementable?

Characterization of GOs

MPQ

- Jamiołkowski Isomorphism [1972]:

physical maps on $\mathcal{B}(\mathcal{H})$ $\leftrightarrow$ states on $\mathcal{H} \otimes \mathcal{H}$

GO G on n modes $\leftrightarrow$ $2n$ mode Gaussian state with CM Γ

- **effect of G_Γ on CM γ:**

$$\gamma \mapsto \Gamma_1 - \Gamma_{12} \frac{1}{\Gamma_2 + \Lambda\gamma\Lambda} \Gamma_{12}^T,$$

where $\Lambda = \text{diag}(1, -1, 1, -1, \ldots)$ and $\Gamma = \begin{pmatrix} \Gamma_1 & \Gamma_{12} \\ \Gamma_{12}^T & \Gamma_2 \end{pmatrix}$.

- preparation of $\rho(\Gamma)$ allows to perform G_Γ

$\Rightarrow$ Gaussian operations are exactly those feasible with linear optics

Summary: Gaussian Operations

- Gaussian operation on n modes: CM Γ on $2n$ modes

- action on CMs

$$\gamma \overset{G_\Gamma}{\longmapsto} \Gamma_2 - \Gamma_{12}^T \frac{1}{\Gamma_1 + \gamma} \Gamma_{12}$$

- contain all transformations from linear optics

- all GO can be implemented with linear optics (including **unlimited squeezing**)

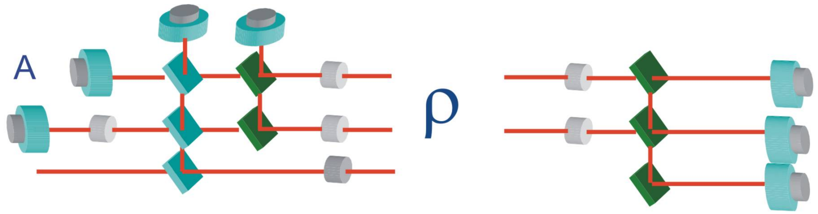
- nonlocal properties of G_Γ:
  - G LOCC iff Γ separable
  - G ppt preserving iff Γ ppt

$\Longrightarrow$ use separability criterion for bipartite Gaussian states

Entanglement Distillation

- given N copies of an entangled Gaussian state ρ

- find Gaussian LOCC to produce maximally entangled state ρ′
- allowed are all Gaussian LOCC, any number of modes, any number of copies of ρ



- previous results indicate: not possible [Eisert et al. 2002; Fiurášek 2002]

No Distillation: Idea of Proof

- define a quantity $V(\gamma)$ related to entanglement such that
  - $V(\gamma_{\text{separable}}) = 1, V(\gamma_{\text{entangled}}) < 1$
  - max. entangled state: $V(\gamma_{\max}) = 0$
  - $V(\gamma \oplus \gamma) = V(\gamma)$ [i.e., $V(\rho \otimes \rho) = V(\rho)$]
- show that Gaussian LOCC cannot decrease V
- use $V(\gamma) := \max \left\{ p \leq 1 : \frac{1}{p}\gamma \quad \text{separable} \right\}$
- G-LOCC have CM $\Gamma = \Gamma_A \oplus \Gamma_B + P, P \geq 0$ [Werner and Wolf, 2001]

$\Rightarrow$ Gaussian LOCC cannot decrease $V(\gamma)$

$\Rightarrow$ **distillation of Gaussian states is not possible with Gaussian operations**

CVs in Atomic Ensembles

- ensemble of N two-level atoms: spin-1/2 formalism

- symmetric states, collective atomic spin

$$J_x := \frac{1}{\sqrt{N}} \sum_{k=1}^{N} \sigma_x^{(i)}, \qquad J_y = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} \sigma_y^{(i)}, \qquad [J_x, J_y] = \frac{i}{N} J_z$$

- z-polarized state: $J_z/N \approx \mathbb{1}$

$\Rightarrow$ $J_x \approx X_A, J_y \approx P_A$ in “polarized subspace”

- light: two polarization modes
  - Stokes parameters: $[S_x, S_y] = iS_z$
  - polarized light: $S_z \approx N\mathbb{1} \Rightarrow \quad S_x/\sqrt{N} \approx X_L, S_y/\sqrt{N} \approx P_L$

- large N, polarized systems: continuous variables appropriate

interaction Hamiltonian $\quad H_{\text{int}} = gX_AX_L$

[Kuzmich et al., 1998; Duan et al., 2001]

What can be done with H_{int}?

- available tools:
  - interaction $H_{\text{int}} = X_A X_L$
  - fast local rotations:

$$H_A = X_A^2 + P_A^2 \qquad H_L = X_L^2 + P_L^2$$

- **strategies:** alternate local rotations V_k and interaction rotate state, apply H_{int} for time t_1, rotate, apply H_{int} for time t_2 etc. until $\sum_k t_k = t$:

$$\mathcal{U}_t = V_n e^{-iH_{\text{int}} t_n} \cdots e^{-iH_{\text{int}} t_2} V_1 e^{-iH_{\text{int}} t_1} V_0$$

  - measurement of X_L (homodyne detection)

Interesting Tasks

- state engineering
  - generate entanglement
  - generate squeezing
  - generate any desired state
- engineering time-evolution
  - Hamiltonian simulation: use H_{int} and H_{loc} to let system evolve according to a desired Hamiltonian H_{eff}
  - gate engineering: realize any desired $\mathcal{U}$
- optimality

Optimal Entanglement Generation

- pure two-mode Gaussian states, CM γ
- smallest eigenvalue $\lambda_{\min}$ of γ bounds entanglement:

$$E_{\text{neg}}(\gamma) \leq 1/\sqrt{\lambda_{\min}}$$

[Wolf et al. 2002]

- applying H_{int} for time t can increase $1/\lambda_{\min}$ at most by factor e^t
- an optimal strategy for $\gamma = \mathbb{1}$ (vacuum): alternate H_{int} for time Δt and local flip $X \rightarrow P$ optimum achieved for $\Delta t \rightarrow 0$

$$\mathcal{U}_t^{\text{opt}} = \lim_{\Delta t \rightarrow 0} \left(V_{\text{flip}} e^{i\Delta t H_{\text{int}}}\right)^{t/\Delta t} = e^{it(X_A P_L + P_A X_L)/2}$$

- measurements and (unsqueezed) ancillas do not help
- squeezed initial states are better

Further results

- engineering of time-evolutions
  - **all Gaussian unitaries can be realized**
  - in particular: U_{swap} which exchanges state of atoms and light: quantum memory (interaction time $t = \pi$ needed)
  - all Hamiltonians $H_{\mathrm{eff}} = aX_AX_L + bX_AP_L + cP_AX_L + dP_AP_L$ can be simulated efficiently
- state engineering
  - can reach all Gaussian states
  - **optimal rate** of entanglement creation for arbitrary pure two-mode state
  - can create spin squeezed atoms without measurement
  - optimal creation of spin squeezing

Summary

- Gaussian operations (GO) on n mode system
  - characterized by $2n \times 2n$ correlation matrix
  - Gaussian LOCC can be identified
  - no distillation of Gaussian states with Gaussian operations

- atom-light–interaction $X_A X_L$
  - can engineer all (Gaussian) unitary time-evolutions
  - entanglement generation in the vacuum state:
    optimal strategy: alternate H_{int} and $X \longrightarrow P$ flip
    best entanglement after time t: $E_{\mathrm{neg}}(t) = e^t$